FLOW OF A VISCOUS FLUID AROUND A SPHERE OF FINITE RADIUS

Yu. G. Ovseenko

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 2, pp. 106-109, 1968

Abstract: We consider the steady flow of a viscous fluid around a sphere of finite radius in non-linear formulation. The equations of motion are written in non-dimensional terms. We seek their solution as the expansion of the unknown stream function in a series of powers of the Reynolds number, the coefficients of which are polynomials in associated Legendre functions of the first kind. Recurrence relations are given for the sequential determination of all coefficients. The velocity and pressure fields are determined. The drag is calculated. Numerical calculations are carried out.

1. Formulation of the problem. Suppose that an incompressible viscous fluid flows around a sphere of radius r_1 where the flow is defined not at infinity as in the classical case but on a sphere of finite radius r_2. In a spherical system of coordinates ($\theta = 0$ is the axis of symmetry), the boundary conditions of the problem are

$$v_r = v_\theta = 0 \quad \text{for} \quad r = r_1,$$

$$v_r = u\cos\theta, \quad v_\theta = -u\sin\theta \quad \text{for} \quad r = r_2. \tag{1.1}$$

We can write the exact Navier-Stokes equations and limiting conditions (1.1) in non-dimensional form:

$$\frac{\partial Q}{\partial x} = -\frac{1}{R}\frac{1}{x^2}\frac{\partial L\Phi}{\partial \tau} + \frac{L\Phi}{x^2(1-\tau^2)}\frac{\partial \Phi}{\partial x},$$

$$\frac{\partial Q}{\partial \tau} = \frac{1}{R}\frac{1}{1-\tau^2}\frac{\partial L\Phi}{\partial x} + \frac{L\Phi}{x^2(1-\tau^2)}\frac{\partial \Phi}{\partial \tau}, \tag{1.2}$$

or, elininating Q:

$$LL\Phi = R\left[\frac{1-\tau^2}{x^2}\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial\tau}\frac{L\Phi}{1-\tau^2} - \frac{\partial\Phi}{\partial\tau}\frac{\partial}{\partial x}\frac{L\Phi}{x^2}\right]; \tag{1.3}$$

$$\frac{\partial\Phi}{\partial\tau} = \frac{\partial\Phi}{\partial x} = 0 \quad (x=1), \qquad \frac{\partial\Phi}{\partial\tau} = -\tau a^2,$$

$$\frac{\partial\Phi}{\partial x} = (1-\tau^2)\,a \qquad (x=a),$$

$$L = \frac{\partial^2}{\partial x^2} + \frac{1-\tau^2}{x^2}\frac{\partial^2}{\partial\tau^2}, \qquad x = \frac{r}{r_1},$$

$$a = \frac{r_2}{r_1}, \quad \tau = \cos\theta, \quad R = \frac{r_1 u}{\nu}. \tag{1.4}$$

Here R is the Reynolds number, $\rho u^2 p(x,\tau)$ is the hydrodynamic pressure, $r_1^2 u\Phi(x,\tau)$ is the stream function, in terms of which the velocity components $v_r = uv(x,\tau)$ and $v_\theta = uw(x,\tau)$ are expressed as

$$v = -\frac{1}{x^2}\frac{\partial\Phi}{\partial\tau}, \qquad w = -\frac{1}{x\sqrt{1-\tau^2}}\frac{\partial\Phi}{\partial x},$$

$$Q = \frac{v^2+w^2}{2} + p.$$

2. Solution of the problem. We seek the solution of problem (1.3) in the form of a series of positive powers of the Reynolds number R

$$\Phi(x,\tau) = \Big(\sum_{k=1}^{\infty} R^{2k-2}\sum_{i=1}^{k}\psi_{k,i}(x)\,P_{2i-1}^1(\tau) +$$

$$\sum_{k=1}^{\infty} R^{2k-1}\sum_{i=1}^{k}\Phi_{k,i}(x)\,P_{2i}^1(\tau)\Big)\sqrt{1-\tau^2}. \tag{2.1}$$

Here the $p_n^1(\tau)$ are the associated Legendre functions of the first kind, we know that (2.1), together with its derivatives up to an order determined by (1.3), converges for small Reynolds numbers [1, 2].

Having substituted the value of the stream function (2.1) into (1.3) and (1.4), and after identifying coefficients of like powers of the Reynolds number on the left- and right-hand sides and then for like $p_n(\tau)$, we obtain an infinite sequence of system of Euler differential equations, with the corresponding boundary conditions

$$G_iG_i\psi_{k,i} = \varphi_{k,i}(x), \quad D_iD_i\Phi_{k,i} = \eta_{k,i}(x) \qquad (k=1,2,\ldots;\ i=1,2,\ldots,k), \tag{2.2}$$

for x = 1,

$$\psi_{k,i} = \frac{d\psi_{k,i}}{dx} = \Phi_{k,i} = \frac{d\Phi_{k,i}}{dx} = 0 \quad (k \geqslant 1);$$

for x = a,

$$\psi_{11} = -\frac{a^2}{2}, \quad \frac{d\psi_{11}}{dx} = -a, \quad \psi_{k,i} = \frac{d\psi_{k,i}}{dx} = 0 \quad (k \geqslant 2),$$

$$\text{and}\quad \Phi_{k,i} = \frac{d\Phi_{k,i}}{dx} = 0 \quad (k \geqslant 1). \tag{2.3}$$

Here,

$$G_i = \frac{d^2}{dx^2} - \frac{2i(2i-1)}{x^2}(\cdots), \qquad D_i = \frac{d^2}{dx^2} - \frac{2i(2i+1)}{x^2}(\cdots);$$

$$\varphi_{k,i}(x) = \sum_{m=1}^{k-1}\sum_{\gamma=1}^{m}\sum_{n=1}^{k-m}\Big\{-\frac{2}{x^2}(\psi'_{k-m,n}D_\gamma\Phi_{m,\gamma}\alpha_2 +$$

$$+\Phi'_{k-m,n}G_\gamma\psi_{m,\gamma}\alpha_4) + 2\gamma\Big[\frac{2\gamma-1}{x^2}\psi'_{k-m,n}D_\gamma\Phi_{m,\gamma} -$$

$$-(2\gamma+1)\Phi_{m,\gamma}\frac{d}{dx}\Big(\frac{G_n\psi_{k-m,n}}{x^2}\Big)\Big]\alpha_1 +$$

$$+(2\gamma-1)\Big[\frac{2\gamma-2}{x^2}\Phi'_{k-m,n}G_\gamma\psi_{m,\gamma} -$$

$$-2\gamma\psi_{m,\gamma}\frac{d}{dx}\Big(\frac{D_n\Phi_{k-m,n}}{x^2}\Big)\Big]\alpha_3\Big\}; \tag{2.4}$$

$$\eta_{k,i}(x) = \sum_{m=1}^{k}\sum_{\gamma=1}^{m}\sum_{n=1}^{k-m+1}\Big\{-\frac{2}{x^2}\psi'_{k-m+1,n}G_\gamma\psi_{m,\gamma}\alpha_6 +$$

$$+(2\gamma-1)\Big[\frac{2\gamma-2}{x^2}\psi'_{k-m+1,n}G_\gamma\psi_{m,\gamma} -$$

$$-2\gamma\psi_{m,\gamma}\frac{d}{dx}\frac{G_n\psi_{k-m+1,n}}{x^2}\Big]\alpha_5\Big\} +$$

$$+\sum_{m=1}^{k-1}\sum_{\gamma=1}^{m}\sum_{n=1}^{k-m}\Big\{-\frac{2}{x^2}\Phi'_{k-m,n}D_\gamma\Phi_{m,\gamma}\alpha_8 +$$

$$+2\gamma\Big[\frac{2\gamma-1}{x^2}\Phi'_{k-m,n}D_\gamma\Phi_{m,\gamma} -$$

$$-(2\gamma+1)\Phi_{m,\gamma}\frac{d}{dx}\frac{D_n\Phi_{k-m,n}}{x^2}\Big]\alpha_7\Big\};$$

$$\alpha_1 = (4l-3)\sum_{p=0}^{\min(2\gamma,\,2l-2)}\frac{a_{2l-p-2}a_{2\gamma-p}a_p}{(4l+4\gamma-2p-3)\,a_{2l+2\gamma-p-2}} -$$

$$(l = i-\gamma+p,\ 1 \leqslant l \leqslant n)$$

$$-(4l-3)\sum_{p=0}^{\min(2\gamma,\,2l-2)}\frac{a_{2l-p-2}a_{2\gamma-p}a_p}{(4l+4\gamma-2p-3)\,a_{2l+2\gamma-p-2}}$$

$$(l=i-\gamma+p+1,\ 1\leqslant l\leqslant n)$$

$$\left(a_j=\frac{(2j-1)!!}{j!}\right) \tag{2.5}$$

The remaining α_i (i = 2, 3, . . . , 8) are defined by similar expressions.

The solution of Eq. (2.2), satisfying (2.3), is

$$\psi_{11}(x)=\frac{a}{2(a-1)^3(4a^2+7a+4)}\times$$

$$\times[-2a^2(a^2+a+1)x^{-1}+6(a^4+a^3+a^2+a+1)x-$$

$$-(4a^4+4a^3+4a^2+9a+9)x^2+3(a+1)x^4]; \tag{2.6}$$

$$\psi_{k,i}(x)=\frac{1}{\Delta_1}\begin{vmatrix}A_{k,i}(x) & x^{1-2i} & x^{3-2i} & x^{2i} & x^{2i+2}\\ A_{k,i}(1) & 1 & 1 & 1 & 1\\ A'_{k,i}(1) & 1-2i & 3-2i & 2i & 2i+2\\ A_{k,i}(a) & a^{1-2i} & a^{3-2i} & a^{2i} & a^{2i+2}\\ A'_{k,i}(a) & (1-2i)a^{-2i} & (3-2i)a^{2-2i} & 2ia^{2i-1} & (2i+2)a^{2i+1}\end{vmatrix};\quad (k\geqslant 2)$$

$$\Delta_1=4a^{4i+1}-(4i-1)^2a^4+2(4i+1)(4i-3)a^2-(4i-1)^2+4a^{3-4i};$$

$$A_{k,i}(x)=\frac{1}{2(4i-1)(4i+1)}\left[x^{2i+2}\int_a^x\xi^{1-2i}\varphi_{k,i}(\xi)\,d\xi-x^{1-2i}\int_1^x\xi^{2i+2}\varphi_{k,i}(\xi)\,d\xi\right]+$$

$$+\frac{1}{2(4i-1)(4i-3)}\left[x^{3-2i}\int_1^x\xi^{2i}\varphi_{k,i}(\xi)\,d\xi-x^{2i}\int_a^x\xi^{3-2i}\varphi_{k,i}(\xi)\,d\xi\right]; \tag{2.7}$$

$$\varphi_{k,i}(x)=\frac{1}{\Delta_2}\begin{vmatrix}B_{k,i}(x) & x^{-2i} & x^{2-2i} & x^{2i+1} & x^{2i+3}\\ B_{k,i}(1) & 1 & 1 & 1 & 1\\ B'_{k,i}(1) & -2i & 2-2i & 2i+1 & 2i+3\\ B_{k,i}(a) & a^{-2i} & a^{2-2i} & a^{2i+1} & a^{2i+3}\\ B'_{k,i}(a) & -2ia^{-2i-1} & (2-2i)a^{1-2i} & (2i+1)a^{2i} & (2i+3)a^{2i+2}\end{vmatrix};\quad (k\geqslant 1)$$

$$\Delta_2=4a^{4i+3}-(4i+1)^2a^4+2(4i+3)(4i-1)a^2-(4i+1)^2+4a^{1-4i};$$

$$B_{k,i}(x)=\frac{1}{2(4i+1)(4i+3)}\left[x^{2i+3}\int_a^x\xi^{-2i}\eta_{k,i}(\xi)\,d\xi-x^{-2i}\int_1^x\xi^{2i+3}\eta_{k,i}(\xi)\,d\xi\right]+$$

$$+\frac{1}{2(4i+1)(4i-1)}\left[x^{2-2i}\int_1^x\xi^{2i+1}\eta_{k,i}(\xi)\,d\xi-x^{2i+1}\int_a^x\xi^{2-2i}\eta_{k,i}(\xi)\,d\xi\right]. \tag{2.8}$$

Equations (2.1) and (2.4)–(2.8) fully define the stream function and therefore the velocity field of the flow under consideration.

In determining the drag of the sphere we will use the value of the function $\psi_{21}(x)$ which is easily obtained by successive application of (2.4)–(2.8).

3. **Determination of the pressure.** If we substitute into (1.2) the expression for the stream function (2.1), integrate the second equation of (1.2) with respect to τ, and then substitute the result into the first equation of (1.2), for the determination of the arbitrary dependence on x (here, it is necessary to take into account that the functions $\psi_{k,i}(x)$ and $\Phi_{k,i}(x)$ are the solutions of Eq. (2.2), we thus establish that the hydrodynamic pressure has the following form:

$$p(x,\tau)=\text{const}+\sum_{k=1}^{\infty}R^{2k-3}\sum_{i=1}^{k}q_{k,i}(x)P_{2i-1}(\tau)+$$

$$+\sum_{k=1}^{\infty}R^{2k-2}\sum_{i=0}^{k}p_{k,i}(x)P_{2i}(\tau),$$

where $P_n(\tau)$ is a Legendre polynomial and the functions $q_{k,i}(x)$ and $p_{k,i}(x)$ are expressed in terms of the coefficients of series (2.1) by complicated formulas which we shall not quote. We simply note that, due to boundary conditions (2.3),

$$q_{k,1}(1)=-\frac{d^3\psi_{k,1}(1)}{dx^3}. \tag{3.2}$$

4. **Frontal drag.** In view of the symmetry of the pressure, the resulting action of the fluid on the sphere is defined by a force directed along the axis of symmetry

$$F=\iint_{(s)}(p_{rr}\cos\theta-p_{r\theta}\sin\theta)\,|_{x=1}\,ds, \tag{4.1}$$

where

$$p_{rr}=\rho u^2\left[-p-\frac{2}{R}\frac{\partial}{\partial\tau}\left(\frac{1}{x^2}\frac{\partial\Phi}{\partial\tau}\right)\right],$$

$$p_{r\theta}=\frac{\rho u^2}{R}\left[-\frac{x}{\sqrt{1-\tau^2}}\frac{\partial}{\partial x}\left(\frac{1}{x^2}\frac{\partial\Phi}{\partial x}\right)+\frac{\sqrt{1-\tau^2}}{x^3}\frac{\partial^2\Phi}{\partial\tau^2}\right]. \tag{4.2}$$

Substituting (4.2) into (4.1), replacing $p(x,\tau)$ and $\Phi(x,\tau)$ by their values from (3.1) and (2.1), taking note of the values of the integrals

$$\int_{-1}^{1}(1-\tau^2)P_n'(\tau)\,d\tau=\begin{cases}4/3, & n=1\\ 0, & n\neq 1,\end{cases}$$

$$\int_{-1}^{1}\tau P_n(\tau)\,d\tau=\begin{cases}2/3, & n=1\\ 0, & n\neq 1\end{cases}$$

and taking note also of the property of the functions $q_{k,1}(x)$ in (3.2), we obtain

$$F=\frac{4\pi\mu r_1 u}{3}\sum_{k=1}^{\infty}R^{2k-2}\left[\frac{d^3\psi_{k,1}(1)}{dx^3}-2\frac{d^2\psi_{k,1}(1)}{dx^2}\right]. \tag{4.3}$$

Retaining only the first two terms of (4.3) and using the values of the functions $\Psi_{11}(x)$ and $\Psi_{21}(x)$, we can find the magnitude of the drag to an accuracy of R^2:

$$F = 6\pi\mu r_1 u\delta_0 \left[1 + \frac{(a-1)\,\delta_1 - 1080\delta_2 \ln a}{\delta_3} R^2\right]. \tag{4.4}$$

Here

$$\delta_0 = \frac{4a\,(a^5-1)}{(a-1)^4\,(4a^2+7a+4)};$$

$$\delta_1 = a^2\,(2160a^{19} + 43\,032a^{18} + 240\,603a^{17} + $$
$$+ 820\,188a^{16} + 2\,027\,184a^{15} +$$
$$+ 4\,101\,096a^{14} + 7\,176\,300a^{13} + 10\,982\,037a^{12} +$$
$$+ 14\,748\,732a^{11} + 17\,662\,944a^{10} +$$
$$+ 19\,012\,953a^9 + 18\,237\,504a^8 + 15\,448\,392a^7 +$$
$$+ 11\,550\,072a^6 + 7\,521\,360a^5 +$$
$$+ 4\,099\,791a^4 + 1\,759\,164a^3 + 544\,968a^2 + 97\,008a + 4512);$$

$$\delta_2 = a^3\,(16a^{18} + 112a^{17} + 448a^{16} + 1260a^{15} +$$
$$+ 2772a^{14} + 5144a^{13} + 8360a^{12} +$$
$$+ 11\,956a^{11} + 15\,116a^{10} + 17\,111a^9 +$$
$$+ 17\,409a^8 + 15\,744a^7 + 12\,544a^6 +$$
$$+ 8788a^5 + 5317a^4 + 2631a^3 + 984a^2 + 252a + 36);$$

$$\delta_3 = 300\,(a-1)^9\,(4a^2+7a+4)^3 \times$$
$$\times (4a^6 + 16a^5 + 40a^4 + 55a^3 + 40a^2 + 16a + 4).$$

The expression for the drag coefficient

$$f(a) = \frac{F}{6\pi\mu r_1 u},$$

for various values of a in terms of the Reynolds number R are:

$$f(2) = 7.2941(1 + 0.00113R^2);$$
$$f(3) = 2.9754\,(1 + 0.00511R^2);$$
$$f(4) = 2.1049\,(1 + 0.01192R^2);$$
$$f(5) = 1.7558\,(1 + 0.02152R^2);$$
$$f(6) = 1.5714\,(1 + 0.03217R^2);$$
$$f(7) = 1.4582\,(1 + 0.04480R^2);$$
$$f(8) = 1.3820\,(1 + 0.05871R^2);$$
$$f(10) = 1.2862\,(1 + 0.08957R^2);$$
$$f(20) = 1.1264\,(1 + 0.22162R^2);$$
$$f(30) = 1.0810\,(1 + 0.49890R^2);$$
$$f(50) = 1.0471\,(1 + 0.97594R^2).$$

We see that, as a increases, the drag coefficient generally decreases, while its second term, which arises from the calculations of non-linear terms in the pressure equations, increases.

We note that the magnitude of the drag from Eq. (4.4), for small Reynolds numbers and for a = 10, 20, 30, hardly differs from the values obtained by Oseen [3] and Praudman and Pearson [4] for the case of the streamlining of a sphere by a flow which is uniform at infinity.

REFERENCES

1. F. K. G. Odqvist, "Über die Randwertaufgaben der Hydrodynamik Zaber Flüssigkeiten," Math. Zeitschrift., 32, no. 3, 1930.

2. Yu. G. Ovseenko, "Motion of a viscous fluid between two rotating spheres," Izv. VUZ., Matematika, no. 4, 1963.

3. C. W. Oseen, "Über die Stokes'sche Formel und Über eine verwandte Aufgabe in der Hydrodynamik,", Arkiv. Matematik, Astr. og Fys., VI, 29, 1910.

4. I. Praudman and D. Pearson, "Expansions in small Reynolds numbers for problems of flow around a sphere and a circular cylinder," Mekhanika, [Russian translation] 2 (48), 1958.

22 June 1967 Novocherkassk